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LETTER TO THE EDITOR

Upper and lower bounds on dynamic correlations in the Griffiths phase

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Abstract. Dilute ferromagnetic spin systems are considered in the Griffiths phase below the percolation threshold. It is shown that, for relaxational dynamics, the spin autocorrelation functions are bounded both above and below by functions of the form $\exp[-A(\ln t)^{d/(d-1)}]$ (Ising systems) and $\exp(-Bt^{1/2})$ (Heisenberg systems), establishing these as the correct asymptotic forms.

A number of recent papers [1-6] have dealt with the dynamics of random magnetic systems in the 'Griffiths phase', which refers to the temperature regime between the transition temperature for magnetic long-range order in the random system and the highest possible transition temperature allowed in principle by a rare statistical fluctuation of the disorder over the whole system. The latter temperature is called the 'Griffiths temperature' T_G. Thus for a ferromagnet with site or bond dilution, T_G is the critical temperature of the undiluted system; for a ferromagnet with a bounded distribution of exchange interactions, T_G is the critical temperature obtained when all bonds take the maximum value. For a ferromagnet with an unbounded distribution of exchange interactions, the Griffiths phase extends to infinite temperature.

For a random ferromagnet, the results [1, 4, 5]

$$C(t) \sim \exp[-A(\ln t)^{d/(d-1)}] \qquad \text{Ising} \tag{1}$$

$$C(t) \sim \exp(-Bt^{1/2}) \qquad \text{Heisenberg} \tag{2}$$

have been obtained for the asymptotic behaviour of the spin autocorrelation functions of Ising and classical Heisenberg spin systems in the Griffiths phase. In (1) and (2), d is the spatial dimension, and the amplitudes A and B depend on the system parameters (temperature, concentration of missing sites or bonds, etc.). The physics behind (1) and (2) concerns the dominance, as $t \to \infty$, of large regions in which, due to rare statistical fluctuations in the disorder, the exchange interactions have values characteristic of an ordered phase at the given temperature. Because these regions are finite they do relax, but only slowly due to their large size.

While the arguments for (1) and (2) are physically appealing, these results have been convincingly derived only as lower bounds on $C(t)$, and then only in the Ising case. It would be gratifying if the forms (1) and (2) could be shown to provide both upper and lower bounds, thereby establishing them beyond doubt as the correct asymptotic forms. In this letter we demonstrate that upper and lower bounds of the forms (1) and (2) *can* be derived for dilute Ising and Heisenberg ferromagnets in at

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least part of the Griffiths phase, namely below the percolation threshold, where the systems are sufficiently dilute that they consist of finite clusters only.

The basic idea is very simple. The lower bound is derived by considering only those clusters which are compact. These are fewer in number than typical clusters, but relax more slowly, in a calculable manner. The upper bound is obtained by including all clusters, but replacing their lifetimes by those of compact clusters of the same volume.

The first step is to write the average spin autocorrelation function, $C(t) = [\langle \boldsymbol{S}_i(t) \cdot \boldsymbol{S}_i(0) \rangle]$, where $\langle \ldots \rangle$ and $[\ldots]$ indicate thermal and disorder averages respectively, as a sum of contributions from different clusters. This is possible because different clusters relax independently. Let n_i be the probability that a randomly chosen site belongs to a cluster of type i, where i specifies both the size of the cluster, and its shape *relative to the chosen site.* (Alternatively, n_i can be thought of as the number of clusters per site of type i.) If $C_i(t)$ is the autocorrelation function of the chosen site, then

$$C(t) = \sum_i n_i C_i(t). \tag{3}$$

The desired bounds will be obtained by writing inequalities for n_i and $C_i(t)$. We consider lower and upper bounds in turn.

(i) *Lower bounds.* For a ferromagnetic system, (3) is a sum of positive terms, so $C(t)$ is bounded below by any subset of these terms. Since we require only the asymptotic form of $C(t)$, we keep things simple by considering compact clusters of a *given shape,* namely hypercubes. Then the index i specifies only the *size* L (i.e. linear dimension) of the cluster. To be specific we will consider site dilution, although bond dilution requires only trivial modifications. Since the cluster volume is L^d, the probability n_L for a given site to belong to such a cluster is

$$n_L = p^{L^d} = \exp[-L^d \ln(1/p)]. \tag{4}$$

In (4) a factor $(1-p)^{\Lambda}$, where $\Lambda \propto L^{d-1}$ is the surface area of the cluster, has been omitted since it leads to a term of order L^{d-1} term in the exponent, which is negligible compared with the $O(L^d)$ term for large L. For similar reasons, a prefactor L^d, corresponding to the number of ways of positioning the site in the cluster, has been omitted. It remains to compute the autocorrelation function $C_L(t)$ for the spins of such a cluster. To obtain the asymptotic (long-time) behaviour we work at timescales sufficiently long that only a single mode remains unrelaxed, namely that which corresponds to a global reorientation of the cluster magnetisation. This timescale, the 'ergodic time' for the cluster, will be different for Ising and Heisenberg systems.

Ising spins. The longest relaxation time for a hypercubic cluster is associated with the creation of a domain wall in the cluster, dividing regions of opposing magnetisation. The free energy of such a wall is, for large L, $F_w = \sigma L^{d-1}$, where σ is the bulk surface tension at the given temperature. The Arrhenius formula gives the cluster ergodic time as

$$\tau_L = \tau_0 \exp(\sigma L^{d-1}/T). \tag{5}$$

The precise form (in particular, any L dependence) of the pre-exponential factor τ_0 is unimportant, since the asymptotic form of $C(t)$ will be dominated by the exponential term in τ_L. The asymptotic time dependence of the autocorrelation function for the cluster sites is

$$C_L(t) = m^2 \exp(-t/\tau_L) \tag{6}$$

where m is the magnetisation per site in the bulk, provided L is large compared with the bulk correlation length ξ. The full correlation function $C(t)$ satisfies

$$C(t) \geqslant \sum_L n_L C_L \xrightarrow[t\to\infty]{} m^2 \sum_L \exp[-L^d \ln(1/p) - t/\tau_L]. \tag{7}$$

Inserting (5) and evaluating the sum on L by steepest descents for large t (i.e. selecting the single largest term) yields

$$C(t) \geqslant m^2 \exp[-A_{\max}(\ln t)^{d/(d-1)}] \tag{8}$$

where

$$A_{\max} = (T/\sigma)^{d/(d-1)} \ln(1/p). \tag{9}$$

Heisenberg spins. We consider purely relaxational dynamics only ('model A' of Hohenberg and Halperin [7]). Then relaxation of the cluster magnetisation is by 'diffusion of the order parameter'. For a compact cluster of definite shape (a hypercube say), the longest relaxation time (ergodic time) is proportional [4] to the number of sites L^d:

$$\tau_L = \tau_0 L^d \tag{10}$$

where the precise form of τ_0 is again unimportant. Substituting this form in (7), and again performing the L sum by steepest descents, yields

$$C(t) \geqslant m^2 \exp(-B_{\max} t^{1/2}) \tag{11}$$

where

$$B_{\max} = 2[\ln(1/p)/\tau_0]^{1/2}. \tag{12}$$

We now derive upper bounds for $C(t)$. These will have the same form as (8) and (11), but with $A_{\min}$ and $B_{\min}$ replacing $A_{\max}$ and $B_{\max}$.

(ii) *Upper bounds.* To obtain upper bounds on $C(t)$, we group clusters according to the number of sites s which they contain, i.e. their 'volume'. The total number of s clusters per site, for $p < p_c$ and s large, is given by [8]

$$n_s = b(s) \exp(-a(p)s) \tag{13}$$

where $a(p)(<\ln(1/p))$ vanishes at p_c. The pre-exponential factor $b(s)$ is of no interest here—the exponential dependence dominates for large s. The longest relaxation time of an s cluster depends, in general, on its shape as well as its volume. For $p < p_c$ most clusters are ramified, with correspondingly short relaxation times. The relaxation time of a general cluster is, however, bounded above by that of a compact (e.g. hypercubic†) cluster of the same volume:

$$\tau_s \leqslant \tau_0 \exp(\sigma s^{(d-1)/d}/T) \qquad \text{Ising} \tag{14}$$

$$\tau_s \leqslant \tau_0 s \qquad \text{Heisenberg.} \tag{15}$$

Inserting (13)-(15), in (3) (with i replaced by s), and evaluating the sums on s by the

† Strictly, spherical clusters have the longest relaxation time, but this does not change our results in any important way. We use hypercubic clusters merely to avoid cluttering the text with geometrical factors associated with the volume and surface area of a hypersphere!

method of steepest descents for large t, yields

$$C(t) \leqslant m^2 \exp[-A_{\min}(\ln t)^{d/(d-1)}] \qquad \text{Ising} \tag{16}$$

$$C(t) \leqslant m^2 \exp(-B_{\min} t^{1/2}) \qquad \text{Heisenberg} \tag{17}$$

where

$$A_{\min} = (T/\sigma)^{d/(d-1)} a(p) \tag{18}$$

$$B_{\min} = 2(a(p)/\tau_0)^{1/2}. \tag{19}$$

In the above derivations of upper and lower bounds on $C(t)$ we have been a little cavalier with the inequality signs in as much as we have systematically ignored (subdominant) pre-exponential factors: the stated inequalities really apply only at the level of the exponential factors. In order to make precise statements, therefore, we rewrite the inequalities for $C(t)$ in the form

$$\lim_{t\to\infty}[-\ln C(t)/(\ln t)^{d/(d-1)}] = A \qquad \text{Ising spins} \tag{20}$$

$$\lim_{t\to\infty}(-\ln C(t)/t^{1/2}) = B \qquad \text{Heisenberg spins} \tag{21}$$

where

$$A_{\min} \leqslant A \leqslant A_{\max} \tag{22}$$

$$B_{\min} \leqslant B \leqslant B_{\max}. \tag{23}$$

Equations (20)–(23) substantiate the asymptotic forms (1) and (2) for dilute magnets with relaxational dynamics, at least in that part of the Griffiths phase which lies below the percolation threshold. Since $A_{\min}$ and $B_{\min}$ vanish at p_c, the upper bounds become useless for $p > p_c$, although the lower bounds still hold. Simple variational arguments [4–6] suggest, however, that the forms (1) and (2) hold throughout the Griffiths phase and for more general kinds of disorder than simple dilution.

The question of greatest practical importance concerns the time domain over which the *asymptotic* behaviour can be expected to be observed. If the asymptotic forms (1) and (2) only hold when $C(t)$ is very small, then the results are somewhat academic. The results of Jain [9] for the $d = 2$ Ising ferromagnet, and those of Ogielski [10] for the $d = 3$ Ising spin glass, are not particularly encouraging. In both cases the data are well fitted by stretched exponential forms, $C(t) \sim \exp[-(t/\tau)^{\beta}]$, with a temperature-dependent 'stretched exponent' β. Jain's data, however, are at least consistent with (1) for $C(t) < 10^{-2}$. By contrast, data obtained from recent Monte Carlo studies [11] of bond-dilute $d = 3$ Heisenberg ferromagnets are well described by (2) over a wide range of time, and the temperature-independent stretched exponent $\frac{1}{2}$ is verified. Supplementing (2) by a power-law prefactor, with a temperature-independent power, fits the data well over most of the decay of $C(t)$, failing only at very short times. In these fits, only B and the overall prefactor depend on temperature. It is not yet clear why (2) fits the Heisenberg data so much better than (1) fits the Ising data. It may be that much better fits are obtained if (1) is modified so that $\ln t$ is replaced by $\ln(t/\tau)$, with τ a temperature-dependent timescale. Such a form does in fact emerge naturally in the upper and lower bounds derived above (see also [4]), with $\tau \sim \tau_0(T/\sigma)^{d/(d-1)}$. This modification is irrelevant asymptotically, when $\ln t \gg \ln \tau$, and was therefore omitted from (1), (8) and (16). However, it may be important to retain it when analysing Monte Carlo (or experimental) data.

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